THE THEORY OF THREE-DIMENSIONAL HYPERSONIC FLOW AROUND A THIN WING OF ARBITRARY SPAN BY A NONSTATIONARY RELAXING GAS FLOW

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We examine the three-dimensional uniform hypersonic flow around the windward side of a thin wing whose surface shape depends on the time, at the angle of attack α = const. We shall consider the flow in the shock layer to be accompanied by physicochemical transformations and to be relaxational in nature. We assume that the shock layer thickness is proportional to a small parameter ε, equal to the characteristic value of the ratio between the densities at the bow shock front, and we use the method of a thin shock layer [1] for solution of the problem.

1. We first analyze the case of the flow around a small span wing with the following characteristic dimensions: $b = O(\sqrt{\varepsilon})$, $c = O(\varepsilon)$, $L = 1$. Then by using the known asymptotic representations of the stream parameters in the form of series in ε and the complete system of gas motion equations with physicochemical transformations [2] we obtain

$$dw/dt = 0,\ dv/dt = -(1/\rho_0)\partial p/\partial y; \tag{1.1}$$

$$d\rho_0/dt + \rho_0(\partial v/\partial y + \partial w/\partial z) = 0; \tag{1.2}$$

$$dq_n/dt = Q_n(p_0, T_0, q_m),\ m = 1, \ldots, N; \tag{1.3}$$

$$dh_0/dt = 0,\ \rho_0 = p_0\mu(q_m)/RT_0,\ d/dt = \partial/\partial t + \partial/\partial x + v\partial/\partial y + w\partial/\partial z, \tag{1.4}$$

where ρ_0, T_0, h_0, q_m are the fundamental values of the density, temperature, static enthalpy, and the relaxation parameters normalized by the corresponding value behind the compression shock, p_0 is the fundamental ("Newtonian") value of the pressure, and the remianing notation is the same as in [2, 3].

The boundary conditions for the system of equations (1.1)-(1.4) have the following form on the shock front for $y = \Phi(x, z, t)$

$$w = -\Phi_z,\ v = \Phi_t + \Phi_x - \Phi_z^2 - 1,\ p = 2\Phi_x + 2\Phi_t - \Phi_z^2 - 1,\ \rho_0 = T_0 = h_0 = q_m = 1, \tag{1.5}$$

and on the body surface for $y = F(x, z, t)$

$$v = F_x + F_t + wF_z. \tag{1.6}$$

There follows from (1.1) and (1.2)

$$\frac{d}{dt}(\rho_0^{-1}w_y) = 0. \tag{1.7}$$

An analogous result is obtained in [4] for stationary perfect gas flows.

We go over to new variables x, ψ, θ, t (for which $d\psi/dt = d\theta/dt = 0$, $\partial(\psi, \theta)/\partial(y, z) \neq 0$) in (1.1), (1.7) and (1.3):

$$w_t + w_x = 0,\ (\rho_0^{-1}w_y)_t + (\rho_0^{-1}w_y)_x = 0,$$
$$z_t + z_x = w,\ (q_n)_t + (q_n)_x = Q_n(p_0, T_0, q_m).$$

Integrating we obtain

$$w = \varphi(\psi, \theta, \tau),\ \rho_0^{-1}w_y = \chi(\psi, \theta, \tau),\ \tau = x - t,$$
$$z = Z(\psi, \theta, \tau) + x\varphi(\psi, \theta, \tau),\ q_n = q_n(x - x_0),$$
$$\rho_0 = \rho_0(x - x_0),\ T_0 = T_0(x - x_0),\ x_0 = \xi(\psi, \theta, \tau).$$

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Here φ, χ, Z, ξ are arbitrary functions. Assuming without loss of generality, as in [5], that $\varphi = \psi$, $Z = \theta$, and going over to the variables x, ψ, z, t, we find

$$\rho_0\chi(\psi, z - \psi x, \tau) = y_\psi,\ y_t + y_x + \psi y_z = v, \qquad (1.8)$$
$$v_t + v_x + \psi v_z = -\rho_0^{-1}\psi_y p_\psi.$$

Integrating (1.8) and satisfying the boundary conditions (1.5) and (1.6), we obtain the general solution of the boundary value problem (1.1)-(1.6) in the form

$$w = \psi,\quad y = F(x, z, t) + \int_\mu^\psi G(\psi_1, z - \psi_1 x, \tau)\,\rho_0^{-1}(x - x_0)\,d\psi_1, \qquad (1.9)$$
$$v = F_t + F_x + \psi F_z + (\mu - \psi)\,\mu_z G(\mu, z - \mu x, \tau)\,\rho_0^{-1}(\mu, z - \mu x, \tau) +$$
$$+ \int_\mu^\psi [(\psi - \psi_1)(G\rho_0^{-1})_{\theta_1} + G\rho_{0\zeta}^{-1}]\,d\psi_1,$$
$$p = 2\Phi_x + 2\Phi_t - \Phi_z^2 - 1 - \int_\nu^\psi (v_t + v_x + \psi v_z)\,G(\psi_1, z - \psi_1 x, \tau)\,d\psi_1.$$

Here $G = \chi^{-1}$; $\zeta = x - x_0$, where for y = F(x, z, t) on the body surface $\psi = \mu$, $\mu_t + \mu_x + \mu\mu_z = 0$, and for $y = \Phi(x, z, t)$ on the shock surface

$$\psi = \nu,\ \nu = -\Phi_z,\ x = x_0(\nu, z - \nu x, \tau),$$
$$(\nu_t + \nu_x + \nu\nu_z)G(\nu, z - \nu x, \tau) = 1$$

(the last equation follows from the solution of (1.10) for y and the boundary conditions (1.5) for v and ρ_0).

The shape of the bow shock and the functions $G(\psi, z - \psi x, \tau)$, $\xi(\psi, z - \psi x, \tau)$ are found from the following system of equations

$$\Phi(x, z, t) = F(x, z, t) + \int_\mu^\nu G(\psi_1, z - \psi_1 x, \tau)\,\rho_0^{-1}(x - x_0)\,d\psi_1; \qquad (1.10)$$
$$\nu = -\Phi_z,\ (\nu_t + \nu_x + \nu\nu_z)G(\nu, z - \nu x, \tau) = 1,\ \xi(\nu, z - \nu x, \tau) = x. \qquad (1.11)$$

For nonrelaxing gas flows (for ρ = const) the formulas (1.9) go over into the solutions obtained in [3, 5].

The relationships (1.9) remian valid even for an inhomogeneous distribution of the relaxation parameters in the oncoming stream if the quantity $\rho(x - x_0)$ therein is replaced by $\rho(\psi, x, z, t)$.

2. The quantity $G(\psi, \theta, \tau)$ in the solutions (1.9)-(1.11) plays the part of a certain Green's function. Let us analyze its structure. Differentiating the last of the equalities (1.11) with respect to the variables x, z, t and evaluating the determinant Δ of the system of linear inhomogeneous equations

$$\begin{pmatrix} \nu_x - \nu - x\nu_x & -1 \\ \nu_z & 1 - x\nu_z & 0 \\ \nu_t & -x\nu_t & 1 \end{pmatrix}\begin{pmatrix} \xi_\nu \\ \xi_\vartheta \\ \xi_\tau \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

we find

$$\xi_\nu - x\xi_\vartheta = \Delta^{-1} = (\nu_t + \nu_x + \nu\nu_z)^{-1},\ \text{where } \vartheta = z - \nu x.$$

Therefore, we will have on the characteristic manifold

$$G(\psi, \theta, \tau) = \partial x_0/\partial\psi - x_0\partial x_0/\partial\theta,\ x_0 = \xi(\psi, \theta, \tau). \qquad (2.1)$$

The selection of the specific dependence $x_0 = \xi(\psi, \theta, \tau)$ in the relationships (2.1), (1.9)-(1.11) is equivalent to solving the inverse problem of the flow around a wing (when the shape of the body surface is determined by the shape of the shock [6]). Indeed, let

$$x_0 = f(\lambda, \tau),$$

where $\lambda = \theta/\psi$, $\lambda^{(\nu)} = z/\nu - x$, $\lambda^{(\mu)} = z/\mu - x$, $\mid \mu = \gamma(x, \tau)z$ (or $G(\mu, \theta^{(\mu)}, \tau) = 0$ [4]), $\gamma_x + \gamma^2 = 0$ $\theta^{(\mu)} = z - \mu x$. We then obtain on the basis of (1.2)

$$\Phi(x, z, \tau) = F(x, z, \tau) + \int_{\lambda^{(\mu)}}^{\lambda^{(\nu)}} G(\lambda, x, \tau)\, d\lambda \tag{2.2}$$

$$F(x, z, \tau) = \Gamma(x, \tau) - \frac{z^2}{2}\left[\lambda^{(\nu)}(x, \tau) + x\right]^{-1},$$

$$G(\lambda, x, \tau) = f_\lambda(\lambda, \tau)\left[\lambda + f(\lambda, \tau)\right] \frac{\rho_0\left[x - f(\lambda, \tau)\right]}{\lambda + x},$$

$$\nu = \frac{z}{\lambda^{(\nu)}(x, \tau) + x}, \quad F = -\nu - \frac{\partial}{\partial z}\int_{\lambda^{(\mu)}}^{\lambda^{(\nu)}} G(\lambda, x, \tau)\, d\lambda.$$

Here $\lambda^{(\nu)}(x, \tau)$, $\Gamma(x, \tau)$ are arbitrary functions.

In conformity with the solution (2.2), the shock layer thickness for x = const depends only on the time, where (2.2) goes into the known solution for a stationary conical flow [6] for $f(\lambda, \tau) = b\lambda/(1 - \lambda)$, $\mu = b^{-1}z/x$, $b = \text{const}$, $\rho_0 = \text{const}$, $\Gamma = 0$.

3. As is known, the flow around a wing of finite span within the framework of the theory of a thin shock layer contains three characteristic cases, usually examined separately [7-10]:

$$\text{a) } \mu^0 \sim \varphi^0, \quad \text{b) } \mu^0 \ll \varphi^0, \quad \text{c) } \mu^0 \gg \varphi^0,$$

where μ^0 is the Mach cone angle in the compressed layer, and φ^0 is the angle at the wing apex.

Let us note that the case 'a' is generally the most general since the asymptotic representations of the stream parameters, the motion equations, and the boundary conditions follow from the corresponding expressions for the case 'a' for performing the passages to the limit:

$$\text{b) } \partial/\partial z \to 0, \; w \to 0; \quad \text{c) } \partial/\partial x \to 0, \; u \to 0.$$

In this connection, it is natural to expect that the general analytical solution (1.9) obtained for the case 'a' will go continuously over into the corresponding solutions for the cases 'b' and 'c'. Let us show this.

We go over to the variables χ, ω, τ_1 in the solution (1.9):

$$N = \psi = \nu(\chi, \omega, \tau_1), \quad \omega = z - (x - \chi)\nu(\chi, \omega, \tau_1), \tag{3.1}$$

$$\tau_1 = t - (x - \chi), \quad d\psi = \frac{\nu_{\tau_1} + \nu_\chi + \nu\nu_\omega}{1 + (x - \chi)\nu_\omega}\, d\chi;$$
$$x - \chi = \frac{z - \omega}{\nu}, \quad d\psi = \frac{\nu_{\tau_1} + \nu_\chi + \nu\nu_\omega}{\nu\left[1 - (\nu_\chi + \nu_{\tau_1})\frac{z - \omega}{\nu^2}\right]}\, d\omega. \tag{3.2}$$

Here χ, ω are, respectively, the longitudinal and side coordinates of the streamline input into the shock layer [4].

Going over to the variables of integration (3.1) and (3.2), respectively, in the general solution, we obtain*

$$y = F(x, z, t) + \int_{\chi_b}^{\chi} \rho_0^{-1}(x - x_0)\, J(x, x_0, \omega)\, dx_0, \tag{3.3}$$

$$p = p_s - \int_{\chi_b}^{\chi} (D_N v)\, J(x, x_0, \omega)\, dx_0;$$

*Formulas analogous to (3.3) and (3.4) were obtained for the stationary case by A. I. Golubinskii and V. N. Golubkin.

$$\Phi(x, z, t) = F(x, z, t) + \int_{\chi_b}^{\chi} \rho_0^{-1}(x - x_0) J(x, x_0, \omega)\, dx_0, \tag{3.4}$$

$$J(x, x_0, \omega) = 1 + (x - x_0) N_\omega(x_0, \omega, \tau_1)$$

or

$$y = \Phi(x, z, t) - \int_z^{\omega} \rho_0^{-1}\left(\frac{z - z_0}{N}\right) I(x, \chi, z_0)\, dz_0, \tag{3.5}$$

$$p = p_s - \int_z^{\omega} (D_N v) I(x, \chi, z_0)\, dz_0;$$

$$\Phi(x, z, t) = F(x, z, t) - \int_z^{\omega_b} \rho_0^{-1}\left(\frac{z - z_0}{N}\right) I(x, \chi, z_0) N^{-1} dz_0, \tag{3.6}$$

$$I(x, \chi, z_0) = \left[1 - (N_\chi + N_{\tau_1}) \frac{z - z_0}{N^2}\right]^{-1};$$

where

$$z - \omega = (x - \chi) N(\chi, \omega, \tau_1),\ \tau_1 = t - (x - \chi),$$
$$N(\chi, \omega, \tau_1) = -\Phi_z(\chi, z = \omega, \tau_1),\ N_\omega = -\Phi_{zz}(\chi, z = \omega, \tau_1). \tag{3.7}$$

Here $(D_N v) = v_t + v_x + N v_z$; $v = y_t + y_x + N y_z$, and p_s is the magnitude of the pressure on the shock front. The selection of the quantities χ_b, ω_b in the general case depends on the flow conditions over the wing leading edge $z_0 = z_0(x)$ and on the thickness of the vortex sublayer [10], where $p_z = O(1)$. For example, for a shock attached only at the wing apex [7], we will have $\chi_b = 0$, $\omega_b = 0$, while for a shock attached along a smooth leading edge

$$z - \omega_b = (1/2)[\gamma - \beta + \sqrt{(\gamma + \beta)^2 - 4}](x - \chi_b) \quad [11],$$
$$\gamma = z_\chi(\chi_b),\ \beta = F_z(\chi_b, \omega_b, \tau).$$

The solution of the system of equations (3.5), (3.7) or (3.6), (3.7) permits finding the unknown shape of the shock $y = \Phi(x, z, t)$. These equations are more convenient for numerical integration of the direct problem than (1.10) and (1.11).

4. The solution of the direct problem of flow around a wing for case 'b' follows from the system of equations (3.4) and (3.5) for $\partial/\partial\omega \to 0$ and the limit relationships 'b'. In this case analytic expressions in the form

$$v = F_x + F_t + \rho_0^{-1}(x - \chi_0) - \rho_0^{-1}(x - \chi),$$
$$p = p_b + (\chi_0 - \chi) \rho_{0x}^{-1}(x - \chi_0) + \rho_0^{-1}(x - \chi_0) - \rho_0^{-1}(x - \chi), \tag{4.1}$$
$$p_b = 2F_x + 2F_t + (x - \chi_0)(F_{tt} + 2F_{xt} + F_{xx}) + \left[(x - \chi_0) \rho_0^{-1}(x - \chi_0)\right]_x$$

are obtained successfully for the stream parameters. Here p_b is the pressure on the wing surface, $\chi_0(z)$ is the equation of the projection of the wing leading edge on the xOz plane, and the form of the function $\rho_0(x - x_0)$ depends on the form of the functions $q_m(x - x_0)$ satisfying the selected system of relaxing equations (1.3), where

$$y = F(x, z_0, t) + \int_{\chi_0}^{\chi} \rho_0^{-1}(x - \xi)\, d\xi,\quad z = z_0 = \text{const},$$

$$\Phi(x, z_0, t) = F(x, z_0, t) + \int_{\chi_0}^{\chi} \rho_0^{-1}(x - \xi)\, d\xi.$$

For ρ = const the formulas (4.1) agree with the result in [3].

Analogously to the above, the solution of the direct problem for case 'c' follows from the system of equations (3.6) as $\partial/\partial\chi \to 0$ and the limit relations 'c':

$$v = D_N(z_0)\Phi - \int_z^{z_0} D_N(z_0)\left(\frac{\rho_0^{-1}I}{N}\right)d\omega - N(z_0)\Phi_z^{-1}, \tag{4.2}$$

$$p = p_s - \int_z^{z_0}[D_N^2(\omega)y]\,I(\omega, z, \tau_1)N^{-1}(\omega, \tau_1)\,d\omega,$$

$$I(\omega, z, \tau_1) = \left[1 - N_{\tau_1}N^{-2}(z-\omega)\right]^{-1};$$

$$p_b = p_s - \int_z^0\left[D_N^2(\omega)\Phi - D_N(\omega)\int_z^{\omega} D_N(\omega)\left(\frac{\rho_0^{-1}I}{N}\right)d\omega_1 - D_N(\omega)\frac{N(\omega)}{\Phi_z}\right]d\omega. \tag{4.3}$$

Here $D_N(z_0)$, $\tau_1(z_0)$ are an abbreviated notation (in the number of arguments) for the following quantities

$$D_N(z_0) = \frac{\partial}{\partial t} + N[z_0, \tau_1(z_0)]\frac{\partial}{\partial t},$$
$$\tau_1(z_0) = t - (z - z_0)N^{-1}[z_0, \tau_1(z_0)], \tag{4.4}$$
$$y = \Phi(z, t) - \int_z^{z_0}\frac{\rho_0^{-1}[z_0 - \tau_1(z_0)]}{N[z_0, \tau_1(z_0)]}I(z_0, z, t)\,dz_0,$$

z_0 is the coordinate of the streamline input to the shock.

Let us note that the flow in the limit case under consideration is not dependent on x or χ and, consequently, does not, in principle, follow from the relations (3.4). However, because of the presence of a singularity in the solution (4.4) in the vortex sublayer, formulas (4.2)-(4.4) are valid only on the part of the shock layer where $p_z = O(\varepsilon)$. To close the system (4.2)-(4.4), the inner solution in the vortex sublayer must be found analogously to [10] and compared with the outer solution (4.2)-(4.4):

$$N[z_0 \to 0, \tau_1(z_0)] \to w(y_1 \to \infty), \; y_1 = y/\sqrt{\varepsilon}. \tag{4.5}$$

The relationships (4.3) and (4.5) form a closed system of equations to find the shape of the shock $y = \Phi(x, z, t)$.

In the stationary case and the equilibrium state of the vortex sublayer, the relationships (4.3)-(4.5) acquire a simpler form on the flat surface (F = 0)

$$p_b(z) = p_s(z) - [1 - N^{-2}(z)]N_z x(z) + N(z)\rho_{0z}^{-1} + \int_0^z N^{-1}(\omega)x(\omega)\rho_{0zz}^{-1}d\omega, \tag{4.6}$$
$$\left[N(z) + N^{-1}(z) + \int_0^z N^{-1}(\omega)\rho_{0z}^{-1}d\omega\right] = A(\ln N_b)_z.$$

Here $p_b(z) = -\frac{1}{2}(N_b^2+1)\rho_{ef}$; $\rho_{ef} = \rho_e/\rho_f$; $\sqrt{\varepsilon} \gg \Omega$*, Ω is a parameter introduced in [7], $\Omega \to 0$, $\varepsilon \to 0$); ρ_e, ρ_f are, respectively, the equilibrium and frozen values of the density, A = const (the quantity A is determined from the boundary conditions on the wing edge [10]), $x(\omega) = \int_0^{\omega} N(z_0)dz_0$; and the subscript "b" refers to the plate surface. For ρ = const the relationships (4.6) agree with those obtained earlier in [10]

$$p_b = p_s - (1 - N^{-2})N_z\int_0^z N(\omega)\,d\omega,$$
$$N(z) + N^{-1}(z) = A(\ln N_b)_z, \; p_b(z) = -\frac{1}{2}(N_b^2+1), \; p_s = -1 - N^2. \tag{4.7}$$

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*The cases $\sqrt{\varepsilon} \sim \Omega$, $\sqrt{\varepsilon} \ll \Omega$ require individual consideration but only $\sqrt{\varepsilon} \gg \Omega$, $\varepsilon \to 0$, $\Omega \to 0$ corresponds to the case 'c'.

The solution of the system (4.7) reduces to the solution of one nonlinear ordinary differential equation with a singularity (of the "saddle" type) [10]. Hence, obtaining the final expressions for the nonstationary relaxing stream parameters flowing perpendicularly to the wing set up turns out to be considerably more tedious than in case 'b'.

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AN EXACT SOLUTION FOR THE INTERACTION OF A SUPERSONIC WEDGE WITH THE BOUNDARY BETWEEN TWO GASES

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It is fairly complicated to examine the interaction of a moving body with inhomogeneities (shock waves or contact discontinuities) in a gas flow. The problem is a nonlinear nonstationary one, in which there is a series of interactions between the shock waves, contact discontinuities, and expansion waves. Therefore, only the linear formulation has been used in analytic solution in [1-3].

In the general case, the solution can be found only numerically [4-6]. Exact solutions can be found in certain cases. For example, in [7, 8] there are exact solutions for the flow of an incident shock wave around a moving wedge.

Here we derive a class of exact solutions for the interaction of a wedge moving with a supersonic velocity in an ideal gas with the boundary between two gases. The medium is considered nonviscous.

1. We consider a wedge with a semivertex angle θ (Fig. 1) moving with a supersonic velocity q_0 in a medium where the pressure, density, and adiabatic parameter are correspondingly $p_0 = 1$, $\rho_0 = 1$, γ_0; there is incident on the wedge at some angle β to the axis of motion a contact discontinuity DBF, where DB is part of the surface of the discontinuity that has not yet interacted, BF is the new surface of the discontinuity, ABC is the head shock wave, BE is the shock wave reflected from the surface of the contact discontinuity, and φ is the angle formed by the head wave. We examined the flow picture on the upper surface of the wedge subject to the condition that the shock wave BE is reflected from the contact discontinuity. The case with a negative-pressure wave is not considered.

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